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COMPRESSIVE STRENGTH OF TAPERED AIRPLANE STRUTS.

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COMPRESSIVE STRENGTH OF TAPERED AIRPLANE STRUTS.*

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Contents.- Methods are here given for ascertaining the value of n in Euler's simplified formula, $P = n\,\frac{EI}{l^2}$, for the compressive strength of tapered airplane struts, by estimating from curves and by calculation.

I. Approximate Method by Means of a Set of Curves.

The effort to make all parts of airplanes as light as possible, and with the minimum air resistance, leads to the employment of posts or struts with their maximum section in the middle and tapered toward the ends. The question of the best longitudinal section of such struts has already been discussed.** Design and material often do not, however, permit reliance on the shapes therein prescribed and the assumption there set forth, in accordance with the rules of the calculus of variations, that the weight alone, or the head resistance plus a fraction of the weight should reach a minimum, is not completely established and is certainly controvertible, as may be seen from differences in

* From Technische Berichte, Volume III, No.7 (1918), pp. 279-281.
** Zeitschrift für Mathematik und Physik, Volume 62, No.2, "Träger kleinster Durchbiegung und Stäbe grosster Knickfestigkeit bei gegebenem Materiel Verbrauch" ("Girders of Minimum Deflection and Struts of Maximum Compressive Strength for a Given Amount of Material"). Zeitschrift für Flugtechnik und Motorluftschiffahrt, 1918, Nos. 5-6, Kirste: "Das günstigste Längsprofil verjungter Flugzeugstreben" ("The Most Favorable Longitudinal Section for Tapered Airplane Struts").

the papers quoted. It would, therefore, appear more suitable to give the method of calculation for an arbitrarily chosen form of strut. Two methods are here employed.

According to the first method, the course of the moments of inertia, for the cross-sections of the symmetrically built struts, is determined from the center to the end, then divided by the moment of inertia (I_m) of the cross-section at the center, and the curve thus obtained is compared with the set of curves in Fig. 1. When a curve of similar form is found, the characteristic number of the curve is inserted instead of n in the well-known Euler formula

$$P_k = n \frac{E\, I_m}{l^2} \tag{1}$$

for compressive loads and the compressive load P_k is obtained, depending on the variation of the moments of inertia n, the modulus of elasticity E, the moment of inertia of the cross-section at the center of the strut I_m, and the length of the strut l. The compressive loads for the same cross-section in the center are proportional to the numbers, n. A comparison of the curves shows that the values of the characteristic number n, for two curves are approximately equal, when they enclose equal areas.

The differential equation for the line of flexure of a bent rod with variable moment of inertia $J(\xi)$, in which $\xi = \frac{x}{l}$, l = the length of the rod, and x = the distance of any given point from the center of the rod, is

$$E\,I\,(\xi)\,\frac{d^2 y}{d\,\xi^2} + P\,l^2\;y = 0 \tag{2}$$

The solution is $y = f(\xi)$ with the assumption that

$$I(\xi) = -\,\frac{f(\xi)}{f''(\xi)}\,\frac{P\,l^2}{E} \tag{3}$$

If the shape of the line of flexure is taken arbitrarily as $f(\xi)$, then we obtain, from equation (3), for any shape of the line of flexure, the corresponding course of the moments of inertia. If $I(\xi) = I_m\,f(\xi)$ and $f(\xi) = 1$ for $\xi = 0$, we obtain

$$I_m = -\,\frac{f(0)}{f''(0)}\,\frac{P\,l^2}{E} \tag{3a}$$

and if we then assume $-\,\frac{f(0)}{f''(0)} = \frac{1}{n}$, we obtain a compression formula similar to Euler's formula

$$P_k = n\,\frac{E\,I_m}{l^2}$$

The curves shown in Fig. 1 are obtained in this way. The number on each curve is the value of n which must be inserted, together with the moment of inertia of the central section I_m, in equation (1), in order to obtain the compressive load P_k. If curves with the same n are compared, it is obvious that a greater thickness at the center of a strut must be offset by a corresponding reduction toward toward the ends. Regarding the use of the diagram, the example given below is referred to.

II. Calculation Method.

If a series of moments of inertia I_1, I_2, I_3 ... of the strut at the points ξ_1, ξ_2, ξ_3 ... is known, it may be considered as represented by a curve $I(\xi)$. By dividing equation (3) by equation (3a), we obtain

$$i = \frac{I(\xi)}{I_m} = \frac{f(\xi)}{f''(\xi)} : \frac{f(0)}{f''(0)} \tag{4}$$

and with $f(\xi) = 1 + a\xi^2 + b\xi^4 + c\xi^6 \pm \ldots$ (5)

we obtain

$$\frac{f(0)}{f''(0)} = \frac{1}{2a} = -\frac{1}{n} \tag{6}$$

As the first equation of condition, we obtain $f(\xi) = 0$ when $\xi = \frac{1}{2}$, or

$$1 + a\frac{1}{2^2} + b\frac{1}{2^4} + c\frac{1}{2^6} + \ldots\ldots = 0 \tag{7}$$

If $\frac{i}{n} = -k$, then we obtain from equation (4)

$$1 + a(2k + \xi^2) + b(12k\,\xi^2 + \xi^4) + c(30k\,\xi^4 + \xi^6) + \ldots = 0 \tag{8}$$

or, by subtracting equation (7) from equation (8),

$$a\left(2k + \xi^2 - \frac{1}{2^2}\right) + b\left(12k\,\xi^2 + \xi^4 - \frac{1}{2^4}\right) + + c\left(30k\,\xi^4 + \xi^6 - \frac{1}{2^6}\right) + \ldots = 0. \tag{9}$$

We can now form as many equations (9) as there are moments of inertia I_1, I_2, I_3 given and we can find, from equations (7) and (9), a number of unknown quantities (a, b, c ...) one more than the given moments of inertia, e.g. with I_1, I_2, and I_3 given, we can find the quantities d, c, b, a and we finally obtain an equation between a and k, or, if we insert $n = -2a$ and $k = \frac{i}{n}$, we obtain the relation sought between n and i. If (besides I_m) one other moment of inertia is given, we put $c = d = \ldots = 0$ and, in equation (7), from equation (9)

$$b = -a \frac{2k + \xi^2 - \frac{1}{2}^2}{12k\,\xi^2 + \xi^4 - \frac{1}{2}^4}$$

and, with $$a = -\frac{n}{2},\ k = \frac{i}{n},\ i = \frac{I}{I_m},$$

we obtain $$1 - \frac{n}{2^3} + \frac{n}{2} \frac{2\frac{i}{n} + \xi^2 - \frac{1}{2}e}{12\times 2^4 \frac{i}{n}\xi^2 + 2^4\xi^4 - 1} = 0$$

and, from this, $$n = \frac{1-i+24i\xi^2-16\xi^4-\sqrt{(1-i+24i\xi^2-16\xi^4)^2 - -(\xi^2-4\xi^4)\,32-12i\xi^2}}{-4\xi^4} \tag{10}$$

When $\xi = \frac{1}{4}$, we obtain

$$n = 10.67i + 20.0 - \sqrt{113.8i^2 - 85.3i + 400} \tag{11}$$

If two other moments of inertia, I_1 and I_2, are given at the points ξ_1 and ξ_2 we have, for a, b, c, the equations

$$a\left(2k_1 + \xi_1^2 - \frac{1}{2^2}\right) + b\left(12k_1\,\xi_1^2 + \xi_1^4 - \frac{1}{2^4}\right) +$$

$$+ c\left(30k_1\xi_1^4 + \xi_1^6 - \frac{1}{2^6}\right) = 0$$

$$a\left(2k_2 + \xi_2^{\,2} - \frac{1}{2^2}\right) + b\left(12k_2\,\xi_2^{\,2} + \xi_2^{\,4} - \frac{1}{2^4}\right) + c\left(30k_2\,\xi_2^{\,4} + \xi_2^{\,6} - \frac{1}{2^6}\right) = 0$$

$$a\,\frac{1}{2^2} \qquad + b\,\frac{1}{2^4} \qquad + c\,\frac{1}{2^6} \qquad = -1$$

The equation defining n therefore becomes

$$\frac{n}{2}\begin{vmatrix} 2\,\frac{i_1}{n} + \xi_1^{\,2} & 12\,\frac{i_1}{n}\,\xi_1^{\,2} + \xi_1^{\,4} & 30\,\frac{i_1}{n}\,\xi_1^{\,4} + \xi_1^{\,6} \\ 2\,\frac{i_2}{n} + \xi_2^{\,2} & 12\,\frac{i_2}{n}\,\xi_2^{\,2} + \xi_2^{\,4} & 30\,\frac{i_1}{n}\,\xi_2^{\,4} + \xi_2^{\,6} \\ \frac{1}{2^2} & \frac{1}{2^4} & \frac{1}{2^6} \end{vmatrix} =$$

$$= \begin{vmatrix} 12\,\frac{i_1}{n}\,\xi_1^{\,2} + \xi_1^{\,4} - \frac{1}{2^4} & 30\,\frac{i_1}{n}\,\xi_1^{\,4} + \xi_1^{\,6} - \frac{1}{2^6} \\ 12\,\frac{i_2}{n}\,\xi_2^{\,2} + \xi_2^{\,4} - \frac{1}{2^4} & 30\,\frac{i_2}{n}\,\xi_2^{\,2} + \xi_2^{\,6} - \frac{1}{2^6} \end{vmatrix} \qquad (12)$$

The best way to solve equation (12) is by trial: by first inserting two approximate values from the diagram, or ascertaining by means of equation (10) or (11), the correct value of the root according to the rules for approximation, or by reversion ("regula falsi"). It must be observed that here only one particular root is correct in equations (11) and (12). Before the square root of equations (10) and (11) the minus sign must be placed, since, otherwise, the corresponding curve $f(\xi)$ would have negative values.

Example.- Let a Sablatnig strut, 200 cm (78.74 in) long, have a moment of inertia at the center of 70 cm^4 (1.68 in^4). At a distance of 40 cm (15.75 in) from the center the moment of inertia is 62 cm^4 (1.49 in^4) and at a distance of 70 cm (27.56 in), it is 33 cm^4 (.79 in^4). Accordingly, with the above symbols, we have

$$I_m = 70$$

$$i_1 = \frac{62}{70} = 0.885, \qquad \xi_1 = \frac{40}{200} = 0.2$$

$$i_2 = \frac{33}{70} = 0.472, \qquad \xi_2 = \frac{70}{200} = 0.35$$

If the points corresponding to the coordinates i and ξ are plotted in the diagram, the value of n is found to be approximately 8.1. If the two points are joined by a curve, it will be found to intersect the straight line $\xi = \frac{1}{4}$, at $i = 0.765$ and we obtain, from equation (11)

$$n = 10.67 \times 0.765 + 20 - \sqrt{113.8 \times 0.765^2 - 85.3 \times 0.765 + 400}$$

$$n = 8.15.$$

If 8.1 is inserted in equation (12), we obtain:

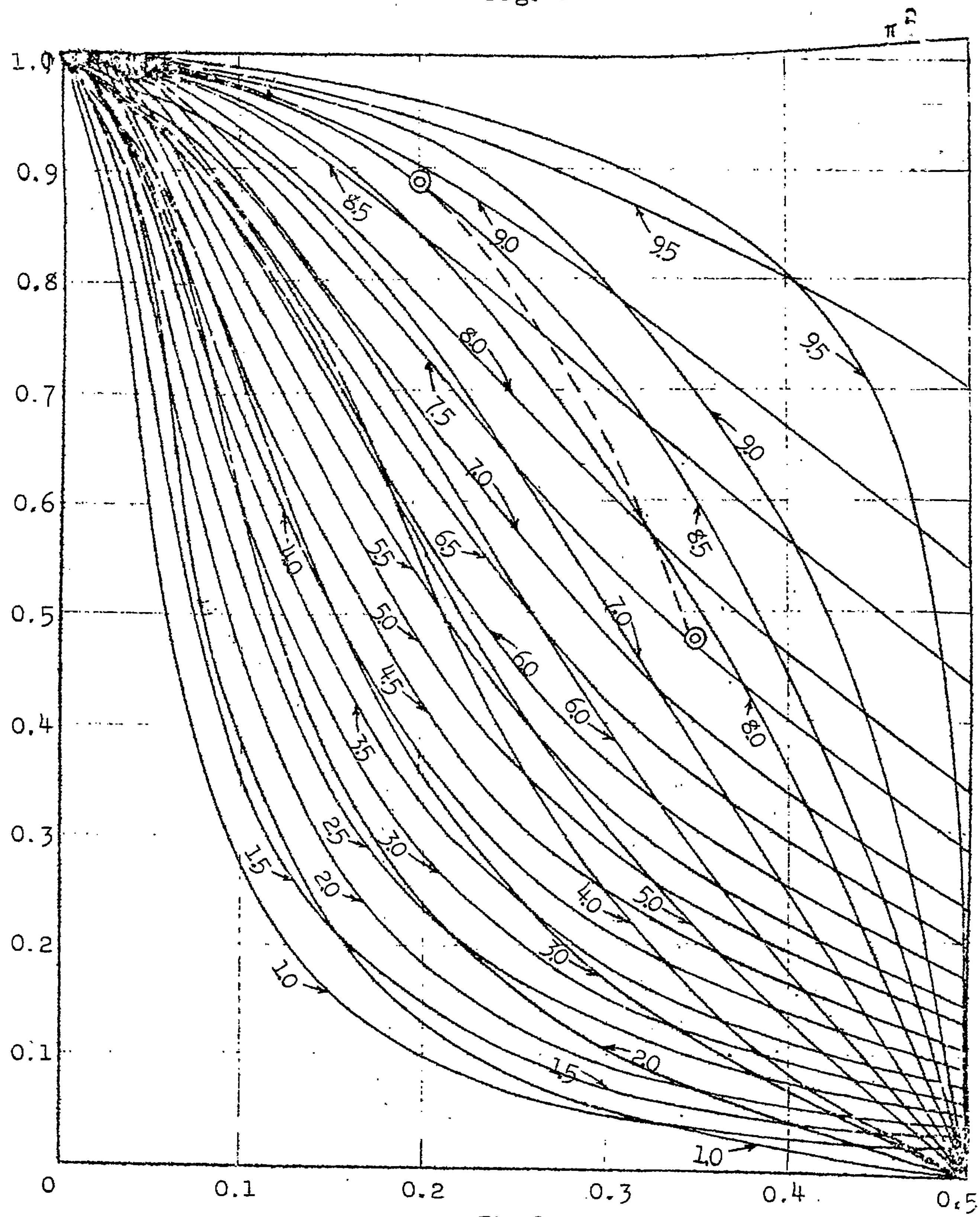
$$4.05 \begin{vmatrix} 0.258 & 0.0532 & 0.00532 \\ 0.2385 & 0.1002 & 0.0320 \\ 0.250 & 0.0625 & 0.0156 \end{vmatrix} - \begin{vmatrix} -0.0085 & 0.0103 \\ 0.0377 & 0.0175 \end{vmatrix} = 0.00002.$$

The value $n = 8.1$ may, therefore, be considered sufficiently exact and we obtain, as the compressive load on the strut,

$$P_k = 8.1 \frac{120000 \times 70}{200^2} = 1700 \text{ kg } (3747.85 \text{ lb}).$$

Translated by National Advisory Committee for Aeronautics.

Fig. 1



π^2
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
9.5
9.0
8.5
8.0
7.5
7.0
6.5
6.0
5.5
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0

Fig.1